J. Fluid Mech. (1983), *vol.* 127, *pp.* 91–101
Printed in Great Britain

On centrifugal separation of a mixture

By H. P. GREENSPAN

Department of Mathematics, Massachusetts Institute of Technology,
Cambridge, Massachusetts 02139

(Received 23 June 1981 and in revised form 13 September 1982)

An exact solution of the equations of motion of a two-phase fluid is given which describes separation in a centrifugal force field. A retrograde rotation is always produced in the settling process, which is characterized by propagating kinematic shocks and a time-dependent volume fraction of the mixture.

1. Introduction

We consider the separation of a fluid mixture in a centrifugal force field. The mixture consists of a dispersed phase of fluid droplets within a continuous phase of another liquid, as for example oil in water (or water in oil). Initially the two-phase fluid occupies the interior of a long cylindrical centrifuge and is in solid-body rotation with the container at angular velocity $\boldsymbol{\Omega}$. Of primary interest is the transient process which leads to the relatively simple, final state of separated fluids in rigid rotation.

Although this problem is the analogue of that considered by Kynch (1952), which dealt with gravitational settling of a mixture, rotational constraints implied by the conservation of angular momentum make for some significant differences. However, as before there is a simple exact solution of the nonlinear equations of motion that govern the rotating two-phase flow.

2. Formulation

Consider a mixture of two incompressible immiscible fluids of constant material properties in which the dispersed phase consists of droplets of approximately uniform size (e.g. spheres of constant radius). In terms of the volume fraction $\alpha = \alpha_D$ of the dispersed phase, the time-averaged variables of velocity $\mathbf{v}_C$, $\mathbf{v}_D$, pressure P_C, P_D and shear stress $\boldsymbol{\tau}_C$, $\boldsymbol{\tau}_D$, the equations of two-phase flow as presented by Ishii (1975) and modified for the rotating coordinate frame are

$$\frac{\partial \alpha}{\partial t} + \nabla . \alpha \mathbf{v}_D = 0, \tag{2.1}$$

$$-\frac{\partial \alpha}{\partial t} + \nabla . (1-\alpha)\mathbf{v}_C = 0, \tag{2.2}$$

$$\alpha\rho_D\left[\frac{\partial \mathbf{v}_D}{\partial t} + \mathbf{v}_D . \nabla\mathbf{v}_D + 2\boldsymbol{\Omega}\times\mathbf{v}_D + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\right] = -\alpha\nabla P_D + \nabla . \alpha\boldsymbol{\tau}_D + \mathbf{M}_D, \tag{2.3}$$

$$(1-\alpha)\rho_C\left[\frac{\partial \mathbf{v}_C}{\partial t} + \mathbf{v}_C . \nabla\mathbf{v}_C + 2\boldsymbol{\Omega}\times\mathbf{v}_C + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\right] = -(1-\alpha)\nabla P_C + \nabla . (1-\alpha)\boldsymbol{\tau}_C + \mathbf{M}_C. \tag{2.4}$$

Here velocities are measured relative to the rotating frame, and gravity is assumed to be negligible. The generalized interfacial drag forces are of the form

$$\mathbf{M}_C = -D(\alpha)(\mathbf{v}_C - \mathbf{v}_D) + \mathbf{M}_C^S, \tag{2.5}$$

$$\mathbf{M}_D = D(\alpha)(\mathbf{v}_C - \mathbf{v}_D) + \mathbf{M}_D^S, \tag{2.6}$$

where $D(\alpha)$ is a drag coefficient. $\mathbf{M}_C^S$, $\mathbf{M}_D^S$ depend on the shear stresses, but it will not be necessary to define these precisely because in the similarity solution to be presented all such shear terms are zero. (However, for definiteness we take $\mathbf{M}_C^S = 0 = \mathbf{M}_D^S$.)

The stress tensors are assumed to depend on the corresponding rates of strains in the conventional manner for Newtonian fluids so that

$$\boldsymbol{\tau}_f = \mu_f(\nabla \mathbf{v}_f + (\nabla \mathbf{v}_f)^\dagger) + \lambda_f(\nabla \cdot \mathbf{v}_f)\mathbf{I} \tag{2.7}$$

for $f = \mathrm{C}$ and $f = \mathrm{D}$. (The extra interfacial deformation tensor that arises from the averaging procedure (Ishii 1975) is taken to be zero in this treatment. Commonly assumed forms would be zero anyway when α is a function of time only, as will be the case here.)

The system of equations is closed by relating the pressures in the two phases through the capillary law

$$P_D = P_C + \frac{2\sigma}{a}, \tag{2.8}$$

where a is the radius of the typical droplet and σ is the interfacial tension.

The mixture occupies the volume between two long concentric cylinders that are the lateral walls of the centrifuge, and on the boundaries $r = r_o, r_i$, $\mathbf{v}_C = \mathbf{v}_D = 0$ for all times. The initial conditions are

$$\alpha = \alpha_I, \text{ a constant,}$$

$$\mathbf{v}_C = 0 = \mathbf{v}_D.$$

The drag coefficient is given by

$$D(\alpha) = \kappa \frac{\mu_C}{a^2} f(\alpha), \tag{2.9}$$

where

$$\kappa = \frac{3}{2}\left(\frac{2\mu_C + 3\mu_D}{\mu_C + \mu_D}\right), \quad \lim_{\alpha \to 0} \frac{f(\alpha)}{\alpha} = 1$$

to allow for the proper reduction to the Rybczinski–Hadamard law in the single-particle limit. Otherwise the form of $f(\alpha)$ is not restricted. For most of this discussion, which deals with fluid–fluid mixtures, we use

$$f(\alpha) = \alpha\left(1 - \frac{\alpha}{\alpha_M}\right) \tag{2.10}$$

and take the maximum volume fraction α_M to be 1. However, for a dispersed phase of solid particles (corresponding to $\mu_D \to \infty$), Barnea & Mizrahi (1973) and Ishii & Chawla (1979) give empirical relationships that imply an 'infinite' drag force at a maximum packing of the sediment.

If the effects of the endwalls can be neglected, about which more will be said, and the major force balance is between centrifugal acceleration and drag, then it may be anticipated that the solution of the problem involves three distinct annular regions.

Let $\rho_D > \rho_C$, so that the dispersed phase is the heavier fluid (unless specifically stated otherwise). There will then be an annular region adjacent to the outer wall, for $R_0(t) \leqslant r \leqslant r_o$, in which the dispersed phase is concentrated to a maximal volume fraction $\alpha = \alpha_M$. For liquid droplets, which are distortable, $\alpha_M = 1$ is a realistic approximation (especially if coalescence can also occur). An inner region $r_i \leqslant r \leqslant R_i(t)$ is occupied by the clarified continuous phase, and the mixture is contained in the narrowing annulus

$$R_i(t) \leqslant r \leqslant R_0(t).$$

Abrupt changes in the flow variables occur across the interfaces $R_i(t)$, $R_0(t)$, which are in fact kinematic shocks.

Although constant conditions prevail in both the condensed and clarified zones, geometrical effects and the variation of centrifugal force with radial distance produce a flow in the mixture layer that is time-dependent. However, the complete solution of the full nonlinear equations can still be obtained by assuming that the velocities are proportional to radial distance, and that the volume fraction is a function of time only. Using dimensionless variables

$$\mathbf{r}_* = \frac{\mathbf{r}}{r_o}, \quad t_* = \frac{\Omega^2|\Delta\rho|a^2}{\kappa\mu_C}t \quad (\Delta\rho = \rho_D - \rho_C),$$

and scales chosen to highlight the balance of the drag force and the effective centrifugal 'gravity' in the rotating frame, a solution is sought of the form

$$\alpha = \alpha(t_*), \tag{2.12}$$

$$\mathbf{v}_C = u_C\hat{\mathbf{r}} + v_C\boldsymbol{\hat\theta} = \left(\frac{|\Delta\rho|a^2}{\kappa\mu_C}\Omega^2 r_o\right)(r_* U_C(t_*)\hat{\mathbf{r}} + r_* V_C(t_*)\boldsymbol{\hat\theta}), \tag{2.13}$$

$$\mathbf{v}_D = u_D\hat{\mathbf{r}} + v_D\boldsymbol{\hat\theta} = \left(\frac{|\Delta\rho|a^2}{\kappa\mu_C}\Omega^2 r_o\right)(r_* U_D(t_*)\hat{\mathbf{r}} + r_* V_D(t_*)\boldsymbol{\hat\theta}), \tag{2.14}$$

$$p_C = \tfrac{1}{2}\rho_C\Omega^2 r_o^2\left(r_*^2 + \frac{|\Delta\rho|}{\rho_C}r_*^2 P(t)\right). \tag{2.15}$$

The substitution of these relationships in (2.1)–(2.4) leads to a set of ordinary differential equations for the unknown time coefficients. Note that all terms that involve the shear stresses $\boldsymbol{\tau}_C$, $\boldsymbol{\tau}_D$ are automatically zero when the velocities vary linearly with radial distance as assumed. Upon dropping the cumbersome asterisk notation, so that hereinafter all variables are dimensionless, the system of equations is

$$\alpha'(t) + 2U_D(t)\,\alpha(t) = 0, \tag{2.16}$$

$$\alpha U_D + (1-\alpha)\,U_C = 0, \tag{2.17}$$

$$\beta(1+\epsilon)\left[\beta|\epsilon|(U_D' + U_D^2 - V_D^2) - 2V_D\right] - \frac{\epsilon}{|\epsilon|} = -P + \frac{1}{\alpha}f(\alpha)(U_C - U_D), \tag{2.18}$$

$$\beta(1+\epsilon)\left[\beta|\epsilon|(V_D' + 2U_D V_D) + 2U_D\right] = \frac{1}{\alpha}f(\alpha)(V_C - V_D), \tag{2.19}$$

$$\beta\left[\beta|\epsilon|(U_C' + U_C^2 - V_C^2) - 2V_C\right] = -P - \frac{1}{1-\alpha}f(\alpha)(U_C - U_D), \tag{2.20}$$

$$\beta\left[\beta|\epsilon|(V_C' + 2U_C V_C) + 2U_C\right] = -\frac{1}{1-\alpha}f(\alpha)(V_C - V_D). \tag{2.21}$$

The two dimensionless parameters

$$\epsilon = \frac{\rho_D - \rho_C}{\rho_C}, \quad \beta = \frac{\Omega a^2}{\kappa \nu_C} \tag{2.22}$$

are the density ratio and a Reynolds number based on particle size and angular velocity.

Equation (2.17), which results from adding (2.1) and (2.2), expresses the fact that there is no radial volume flow, although of course there must be a mass flux in this direction for separation to occur.

The conservation of mass for each component of the mixture applies across the moving interfaces $R_o(t)$, $R_i(t)$. If $\mathscr{U}$ is the (radial) velocity of a surface of discontinuity then

$$\left.\begin{aligned} \alpha(\mathscr{U} - u_D)]_-^+ &= 0. \\ (1-\alpha)(\mathscr{U} - u_C)]_-^+ &= 0. \end{aligned}\right\} \tag{2.23}$$

For example, across the interface $r = R_o(t)$ between the mixture and the maximally condensed dispersed phase we have

$$\mathscr{U} = \frac{dR_o}{dt}, \quad \alpha^+ = \alpha_M, \quad \alpha^- = \alpha, \quad u_D^+ = 0, \quad u_D^- = R_o U_D, \quad u_C^- = R_o U_C. \tag{2.24}$$

It follows that

$$\frac{1}{R_o}\frac{dR_o}{dt} = \frac{(1-\alpha)U_C}{\alpha_M - \alpha} = \frac{-\alpha U_D}{\alpha_M - \alpha}. \tag{2.25}$$

For a fluid–fluid mixture $\alpha_M \approx 1$. In this case the preceding equation becomes

$$\frac{1}{R_o}\frac{dR_o}{dt} = U_C(t). \tag{2.26}$$

(For problems in which solid particles form the dispersed phase, rigid rotation of the condensed sediment with $\alpha_M \neq 1$ is consistent with these equations of motion only if the drag is essentially infinite for $\alpha = \alpha_M$.)

Similarly, across the shock at $r = R_i(t)$, where the continuous phase is purified,

$$\alpha^- = 0, \quad \alpha^+ = \alpha, \quad u_C^- = 0, \quad u_C^+ = R_i U_C, \quad u_D^+ = R_i U_D,$$

so that

$$\frac{1}{R_i}\frac{dR_i}{dt} = U_D = -\frac{1-\alpha}{\alpha} U_C. \tag{2.27}$$

The trajectory $r = R_i(t)$ is then the path of the droplet or particle that was initially in contact with the inner cylinder. If $\alpha_M = 1$, the surface $r = R_o(t)$ also corresponds to the particle path of the element of the continuous phase that was originally in contact with the outer cylinder.

The conservation of total momentum across the kinematic shock provides jump conditions on the pressure and the circumferential velocity components. In terms of the density

$$\rho_m = (1-\alpha)\rho_C + \alpha\rho_D \tag{2.28}$$

and the mass-averaged velocity defined by

$$\rho_m \mathbf{v}_m = (1-\alpha)\rho_C \mathbf{v}_C + \alpha\rho_D \mathbf{v}_D, \tag{2.29}$$

momentum conservation implies in general that

$$[\rho_m \mathbf{v}_m\{(\hat{\mathbf{n}}\,.\,\mathbf{v}_m - \mathscr{U})\} + \hat{\mathbf{n}}p_m - \hat{\mathbf{n}}\,.\,\boldsymbol{\tau}_m]_-^+ = 0. \tag{2.30}$$

Here $\boldsymbol{\tau}_m$ is the mean stress and $\hat{\mathbf{n}}$ is the unit normal to the shock moving with velocity $\mathscr{U}$. For the problems under consideration, the condition on the circumferential component of momentum reduces simply to

$$\mathbf{v}_m]_-^+ = 0. \tag{2.31}$$

In other words, the mean azimuthal velocity component given by (2.29) is continuous across a kinematic shock, which is in effect a no-slip constraint. The jump in the radial momentum component provides a relationship between the pressure on both sides of the shock.

Equations (2.23) may be combined to yield the jump condition for total mass conservation:

$$\rho_m(\mathscr{U} - u_m)]_-^+ = 0. \tag{2.32}$$

The flow in the mixture layer may now be determined (independently from that in the other two annular regions) by integration of the system of equations (2.16)–(2.21), (2.25), (2.27) subject to the initial conditions at $t = 0$,

$$U_C = U_D = V_C = V_D = 0, \quad R_o = 1, \quad R_i = R_I = \frac{r_i}{r_o}, \quad \alpha = \alpha_I. \tag{2.33}$$

The results of these calculations for typical parameter values are presented next.

The flows in the regions of clarified fluid and the condensed discrete phase depend to a large extent on the relative strengths of the effective gravity and the viscous forces which ultimately bring the fluid to rest in the rotating frame. These more conventional spin-up problems of homogeneous fluids are discussed in §4.

3. Solution

Some additional notation is required in order that the results can incorporate both positive and negative values of the density differences $\rho_D - \rho_C$, i.e. $-1 < \epsilon < \infty$. Let $R_D(t)$ and $R_C(t)$ be the interfaces that respectively separate the mixture from the condensed, dispersed phase and the clarified continuous phase. For $\rho_D > \rho_C$, clearly

$$R_D(t) = R_o(t), \quad R_C(t) = R_i(t), \tag{3.1}$$

whereas for $\rho_D < \rho_C$ the reverse holds:

$$R_D(t) = R_i(t), \quad R_C(t) = R_o(t). \tag{3.2}$$

These relations merely reflect the fact that the heavier phase is always thrown outwards. The initial conditions remain $R_o(t) = 1$, $R_i(t) = R_I$. Jump conditions (2.26) and (2.27) are now more generally given by

$$\frac{1}{R_D(t)}\frac{dR_D}{dt} = \frac{1-\alpha}{\alpha_M - \alpha}U_C = -\frac{\alpha}{\alpha_M - \alpha}U_D, \tag{3.3}$$

$$\frac{1}{R_C(t)}\frac{dR_C}{dt} = U_D = -\frac{1-\alpha}{\alpha}U_C. \tag{3.4}$$

These may be combined with (2.16) and (2.17) to obtain $R_D(t)$, $R_C(t)$ as explicit functions of the volume fraction $\alpha(t)$:

$$R_D(t) = R_D(0)\left(\frac{\alpha_M-\alpha_I}{\alpha_M-\alpha}\right)^{\frac{1}{2}}, \tag{3.5}$$

$$R_C(t) = R_C(0)\left(\frac{\alpha_I}{\alpha}\right)^{\frac{1}{2}}. \tag{3.6}$$

Separation is completed when the two fronts meet at time $\bar{t}$, that is $R_D(\bar{t}) = R_C(\bar{t})$ or

$$\bar{\alpha} = \alpha(\bar{t}) = \frac{R_C^2(0)\,\alpha_I\,\alpha_M}{R_D^2(0)\,(\alpha_M-\alpha_I)+R_C^2(0)\,\alpha_I}. \tag{3.7}$$

Since the denser phase is always moved outwards by the centrifugal force, (2.16) implies that the volume fraction of the mixture decreases with time for $\rho_D > \rho_C$ and $U_D > 0$, but increases when $\rho_C > \rho_D$ and $U_D < 0$. The time for complete separation is longest when there is no solid inner surface against which material can collect. For $r_i = 0$, (3.7) indicates that $\bar{\alpha} \to 0$ or $\bar{\alpha} \to \bar{\alpha}_M$, depending on whether ρ_D is less than or larger than ρ_C. Simply put, the volume fraction approaches zero or its maximum as the heavier fluid phase is 'squeezed' out of the mixture.

In order to determine $\alpha(t)$, the remaining equations must be integrated, and this in general can only be done numerically. However, in the limit as $\epsilon \to 0$ (with the step function $s = \epsilon/|\epsilon|$ being ± 1, depending in the sign of ϵ) the equations can still be solved analytically. It is found that

$$U_D = \frac{sf(\alpha)}{\alpha}\left(4\beta^2+\left(\frac{f(\alpha)}{\alpha(1-\alpha)}\right)^2\right)^{-1}, \tag{3.8}$$

$$U_C = -\frac{sf(\alpha)}{1-\alpha}\left(4\beta^2+\left(\frac{f(\alpha)}{\alpha(1-\alpha)}\right)^2\right)^{-1}, \tag{3.9}$$

$$V_C - V_D = 2\beta s\left(4\beta^2+\left(\frac{f(\alpha)}{\alpha(1-\alpha)}\right)^2\right)^{-1}. \tag{3.10}$$

Furthermore, with the specification of the drag coefficient as

$$f(\alpha) = \alpha(1-\alpha), \tag{3.11}$$

and

$$\tau = \frac{t}{(1+4\beta^2)}, \tag{3.12}$$

then

$$\alpha = \frac{\alpha_I e^{-2s\tau}}{1-\alpha_I+\alpha_I e^{-2s\tau}}. \tag{3.13}$$

Moreover, the time of separation is

$$(1+4\beta^2)^{-1}\,\bar{t} = \bar{\tau} = \frac{s}{2}\log\frac{R_0^2(0)\,(\alpha_M-\alpha_I)+\alpha_I\,(1-\alpha_M)\,R_i^2(0)}{\alpha_M\,(1-\alpha_I)\,R_i^2(0)}. \tag{3.14}$$

The initial conditions on velocity cannot be satisfied in this special case. The implication is that for small non-zero ϵ there will be a very short transient phase, a time boundary layer, in which the adjustment of the velocity from the initial setting occurs.

A difficulty in setting $\epsilon = 0$ is that two equations, (2.19) and (2.21), become identical, and for this reason only a formula for the velocity difference $V_C - V_D$

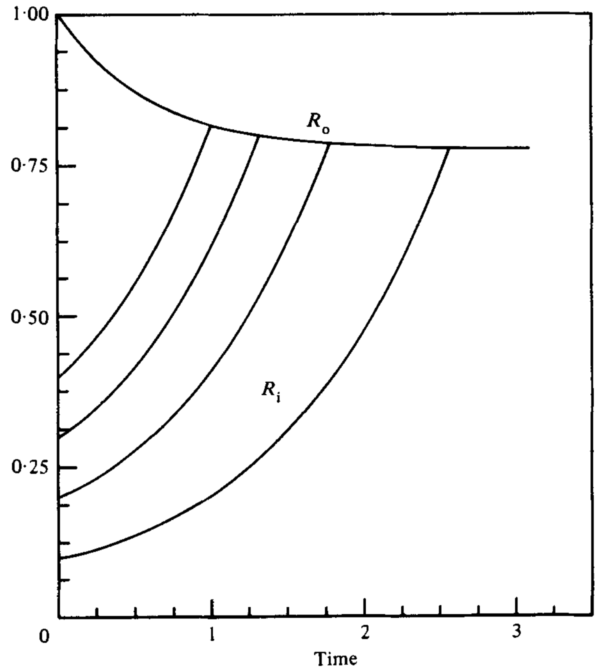

FIGURE 1. Loci of the kinematic shocks $R_o(t)$, $R_i(t)$ that bound the annular region of the mixture for inner radii of the centrifuge $R_i(0) = R_I = 0{\cdot}1, 0{\cdot}2, 0{\cdot}3, 0{\cdot}4$. Parameter setting: $\epsilon = 0{\cdot}1$, $\beta = 0{\cdot}1$, $\alpha_I = 0{\cdot}4$, $\alpha_M = 1{\cdot}0$.

emerges from the reduced system. A formal perturbation expansion in powers of ϵ, although laborious to execute, does lead to individual expressions for V_C and V_D at the next order of approximation. This will not be pursued here; instead, numerical computations for small ϵ are presented to validate the procedure.

The pressure term may be eliminated from the system by subtracting (2.18) from (2.20); the equation obtained is further simplified by using (2.16) and (2.17) to replace $U'_C(t)$. The result of all this algebra is an equation for the derivative of U_D alone:

$$U'_D = \frac{1-\alpha}{1+(1-\alpha)\epsilon}\left[\frac{2\alpha U_D^2}{(1-\alpha)^2} + U_C^2 - V_C^2 - (1+\epsilon)(U_D^2 - V_D^2) \right.$$
$$\left. + \frac{2}{\beta|\epsilon|}((1+\epsilon)V_D - V_C) + \frac{1}{\beta^2|\epsilon|}\left\{\frac{\epsilon}{|\epsilon|} + \frac{f(\alpha)}{\alpha(1-\alpha)}(U_C - U_D)\right\}\right]. \quad (3.15)$$

With the drag coefficient given by (3.11), the final set of equations is then integrated numerically for any particular parameter values of interest. The same procedure can be followed with any other drag law, say one especially for sediment. Indeed the similarity form of solution could apply as well to more complicated drag interactions in which the effects of apparent mass and boundary-layer forces (Zuber 1964) are also included. However, a drag law that is appropriate for high Reynolds numbers in its dependence on the square of the velocity would not preserve similitude.

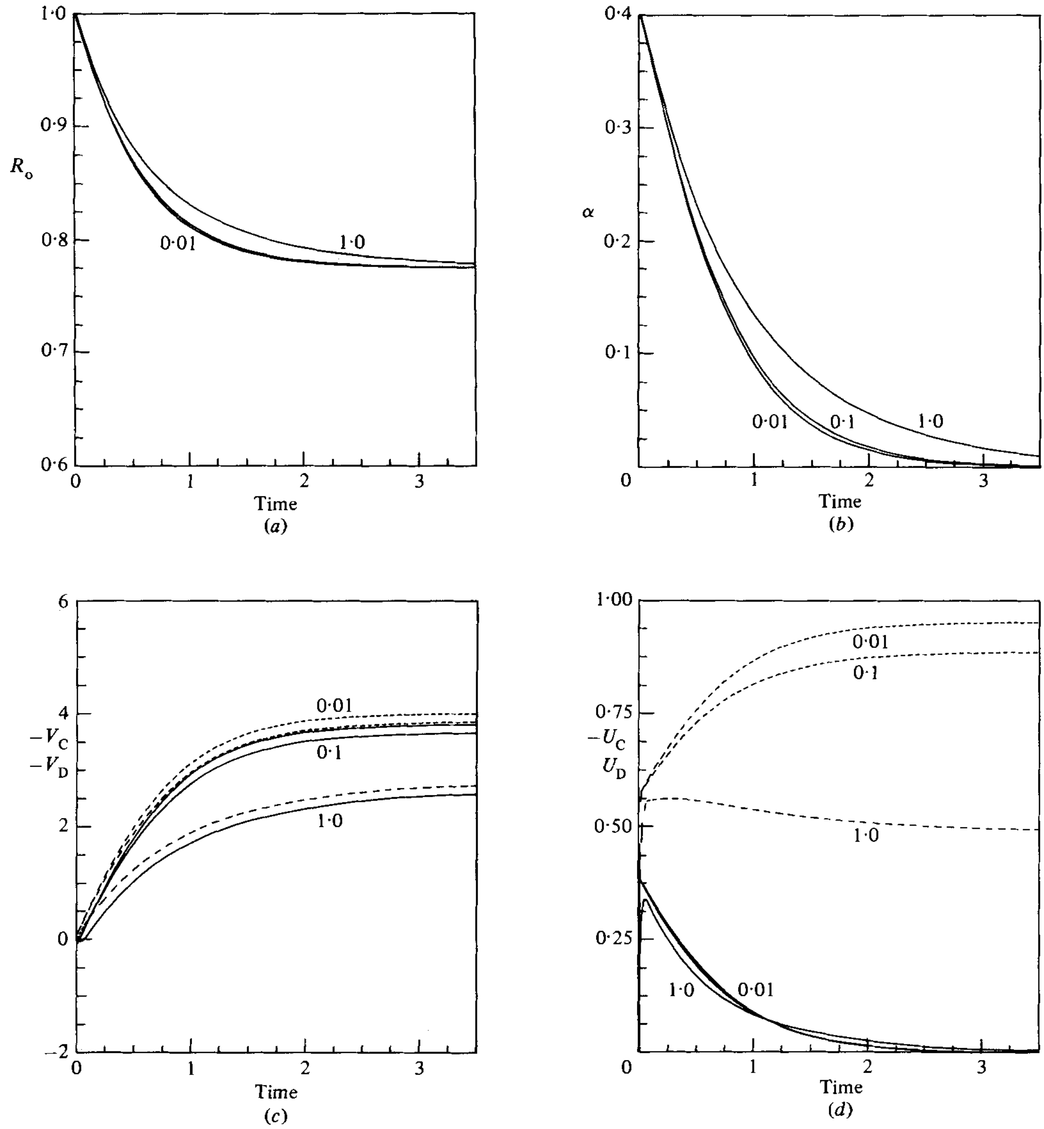

FIGURE 2. Flow variables for $\epsilon = 0{\cdot}01$, $0{\cdot}1$, $1{\cdot}0$ and $\beta = 0{\cdot}1$, $\alpha_I = 0{\cdot}4$, $\alpha_M = 1{\cdot}0$, $R_I = 0$. (*a*) Locus of the kinematic shock from the outer wall. (*b*) Volume fraction versus time. (*c*) Angular velocity versus time. Curves for V_C are solid, those for V_D are dashed. (*d*) Radial velocity coefficients versus time. Curves for U_C are solid, those for U_D are dashed.

4. Discussion

Results of the calculations for parameter values perhaps typical of the fluid–fluid mixture are presented in the figures. Figure 1 shows the loci of the interfaces that separate the mixture layer from the 'purified' or condensed fluid components. Clearly, it is only the position of the shock propagating away from the inside wall that depends on the radius of the interior cylinder. For $r_i = 0$ no such shock forms, since there is no solid surface to serve as a collection plate. In this case the volume fraction of the mixture simply decays to zero uniformly with time.

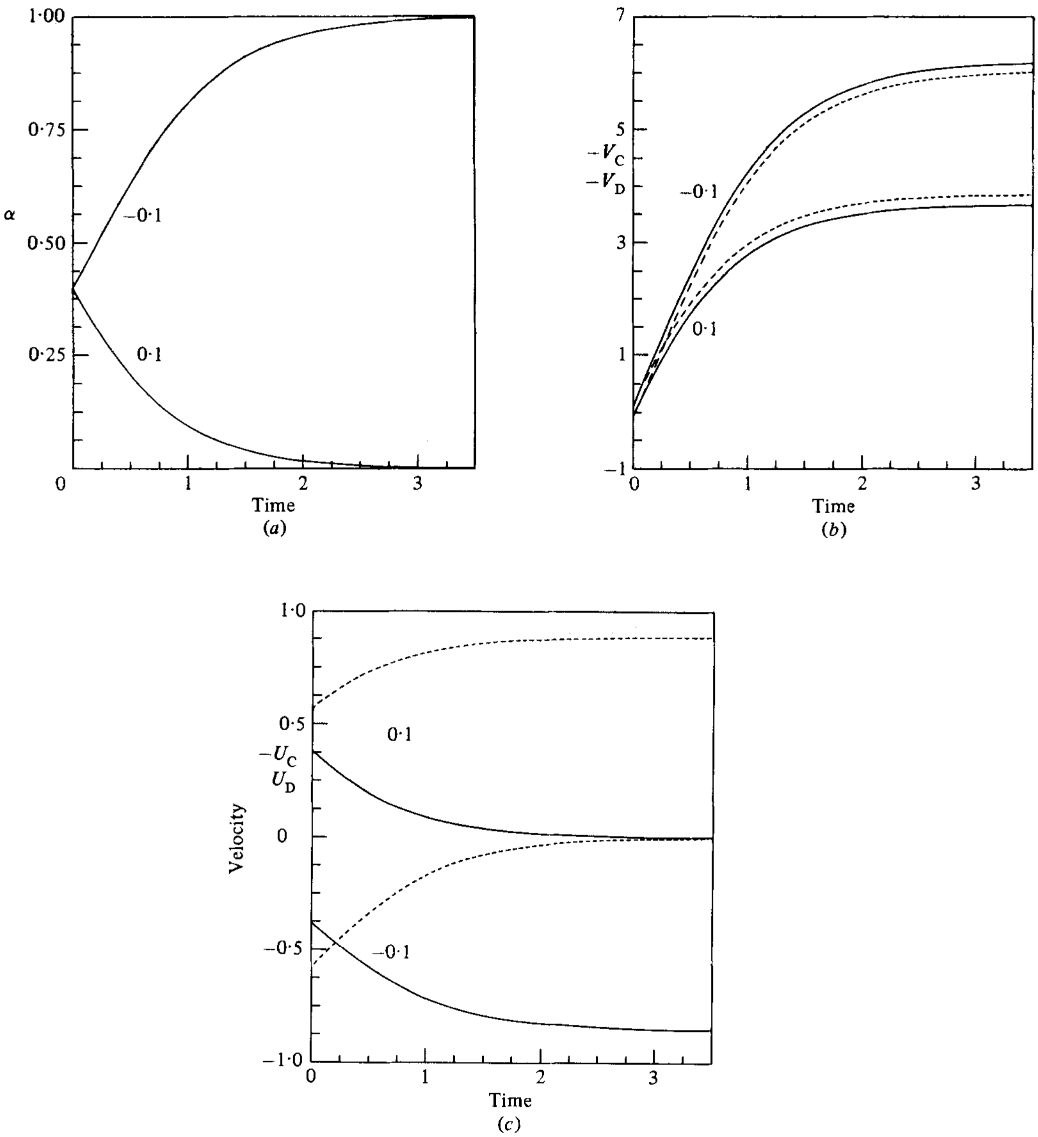

FIGURE 3. Flow variables for $\epsilon = \pm 0{\cdot}1$ and $\beta = 0{\cdot}1$, $\alpha_I = 0{\cdot}4$, $\alpha_M = 1{\cdot}0$, $R_I = 0$. (*a*) Volume fraction versus time. (*b*) Angular velocity versus time. Curves for V_C are solid and those for V_D are dashed. (*c*) Radial velocity coefficients versus time. Curves for U_C are solid and those for U_D are dashed.

For the nominal values $\beta = 0{\cdot}1$, $\alpha_I = 0{\cdot}4$, $\alpha_M = 1$ the dependence of the principal variables on time, for various ϵ, is shown in figures 2 and 3. The dimensionless separation time increases as ϵ increases, partly because of the more effective role of the Coriolis force. For $\epsilon = 0{\cdot}01$ the results are in accord with (3.8)–(3.14). Qualitatively similar results are obtained for drag laws other than (2.10), e.g. $f(\alpha) = \alpha/(1-\alpha)$.

The evolution of $\alpha(t)$ depends on $R_o(t)$ and $R_i(t)$ only to the extent that the meeting of these interfaces sets the time at which separation is completed. This is also true for the velocity components.

An interesting conclusion is that the relative circumferential velocity component of each phase is always negative, which means that separation produces a retrograde

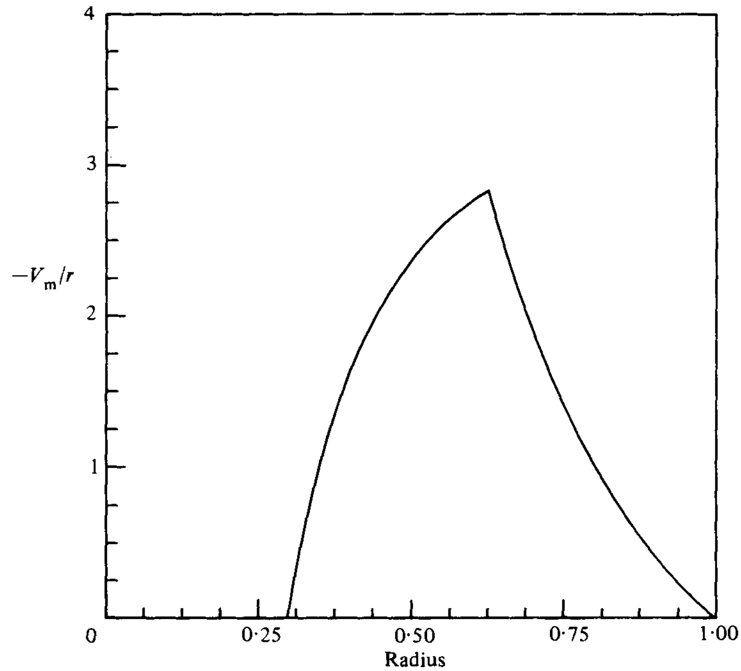

FIGURE 4. Angular velocity of the contained fluid at the instant of completed separation, corresponding to $\epsilon = 0{\cdot}1$, $\beta = 0{\cdot}1$, $\alpha_I = 0{\cdot}4$, $\alpha_M = 0{\cdot}6$, $R_I = 0{\cdot}3$.

rotation with respect to the rotating container. In essence, elements of the heavier fluid phase are displaced the greatest distance in the cylindrical geometry, and their reduced rotational speed, a consequence of the angular-momentum law, confers a lower peripheral speed on the entire fluid through drag interaction.

There is a short transient phase in which accelerations are important and the velocities are adjusted from their initial values to those consistent with the separative process. This time boundary layer will not be discussed in greater detail here.

The flow in the mixture can now be determined for any parameter setting and with modest effort even for more general momentum-interaction terms which include the effects of apparent mass, particle inertia and boundary layers.

It remains to describe the flow of the 'purified' constituents. This depends to a large extent on the relative magnitude of viscous forces, which must eventually restore the entire fluid to a state of rigid rotation. Suppose that the rheological properties of the phases are similar and that the separation time, $O(\nu\rho/\Omega^2a^2\Delta\rho)$, is very short compared to that for fluid spin-up, $O(l/\nu\Omega)^{\frac{1}{2}})$ (l is the length of the centrifuge), or for vorticity diffusion, $O(r_0^2/\nu)$. The position of the shocks and the jump conditions across them are then sufficient to determine the velocities in the annular regions in contact with walls of the centrifuges. The appropriate velocity distribution is that left behind, so to speak, as a front passes a given position. Since the radial velocity components are zero outside the mixture layer and the circumferential components of mean velocity are continuous across the fronts, a plot of v_m/r versus r ($= R_D$) and v_m/r versus r ($= R_C$) (figure 4) gives the angular velocity in the regions of clarified and condensed phases during and after separation. The spin-up of these fluid masses to rigid rotation is accomplished in the spin-up timescale by secondary circulations induced by Ekman layers at the end walls of the centrifuge. The physical process is essentially that described in detail by Greenspan (1968), but modified to account for the two separate fluids, the interface between them, and the vertical boundary layers that must arise at such locations. Of course, endwall effects are always present, and do produce a slight secondary circulation on the basic separating flow described in §3 (which applies ideally only to infinitely long cylinders). If spin-up and settling (or diffusion) times are comparable,

$$\frac{\nu}{\Omega^2 a^2 (\Delta\rho/\rho)} = O\left(\frac{l}{(\nu\Omega)^{\frac{1}{2}}}\right) \quad \text{or} \quad \left(\frac{\Omega}{\nu}\right)^{\frac{3}{2}} a^2 \frac{\Delta\rho}{\rho} l \approx O(1),$$

then the effects of the Ekman layers may be important. The boundary layers must be considered, and the similarity solution modified accordingly when the droplets are extremely small, the rotation rate is low or the density difference is slight.

Finally, in sedimentation problems the kinematic viscosity of the maximally condensed particles is essentially infinite compared with that of the mixture or of the clarified fluid. As solid particles cross the kinematic shock, they are simply added to a sediment layer that is and stays in rigid rotation. However, the fluid motion elsewhere would be as described.

REFERENCES

Barnea, E. & Mizrahi, J. 1973 A generalized approach to the fluid dynamics of particulate systems. *Chem. Engng J.* **5**, 171–189.

Greenspan, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.

Ishii, M. 1975 *Thermo-Fluid Dynamic Theory of Two-Phase Flow*. Paris: Eyrolles.

Ishii, M. & Chawla, T. C. 1979 Local drag laws in dispersed two-phase flow. *Argonne Natl Lab.* NRC FNA 2026.

Kynch, G. J. 1952 A theory of sedimentation. *Trans. Faraday Soc.* **48**, 166–176.

Zuber, N. 1964 On the dispersed two-phase flow in the laminar flow regime. *Chem. Engng Sci.* **19**, 897.